Calculators may be used in this examination provided they are not capable of being used to store alphabetical information other than hexadecimal numbers

UNIVERSITY OF BIRMINGHAM

School of Computer Science

Algorithms and Complexity

Main Summer Examinations 2024

Time allowed: 2 hours

[Answer all questions]

Note

Answer ALL questions. **There are four questions worth 18, 16, 10, 6 marks each for a total of 50 marks.** You can use (without proof) anything that was taught in the module (slides, lectures, weekly assignments).

Question 1

Let $G = (V, E)$ be an undirected graph. Recall that a set of vertices $X \subseteq V$ is said to form a *vertex cover* if at least one endpoint of every edge in E belongs to the set X. The size of a vertex cover $X \subseteq V$ is the number of vertices in X.
Definition 1: A vertex cover $X \subseteq V$ of G is said to be *minimal* if removing any vertex $v \in X$ causes it to not be a vertex cover.
Definition 2: A vertex cover $X' \subseteq V$ of G is said to be *minimum* if the number of vertices in X' is the minimum among all possible vertex covers of G.

Answer the following four questions:

(a) Give an example of a minimal vertex cover which is not a minimum vertex cover. The graph in your example should contain at least 7 vertices. Explain your answer. **[3 marks]**

(b) Design an $n^{O(1)}$ time algorithm that finds a minimal vertex cover in a graph with n vertices. Explain both the correctness and running time of your algorithm. **[5 marks]**

(c) Give an example of a graph G (with at least 10 vertices) where there is a minimal vertex cover whose size is strictly greater than six times the size of a minimum vertex cover of G. Explain your answer. **[4 marks]**

(d) Recall the following two definitions:

- A set of vertices $Y \subseteq V$ is an independent set if no two vertices in Y form an edge in E.
- An independent set $Y \subseteq V$ is maximal if adding any vertex $v \in (V \setminus Y)$ to Y causes it to not be an independent set.

Prove that if Z is a minimal vertex cover then $V \setminus Z$ is a maximal independent set. **[6 marks]**

Question 2

(a) Give an example of an undirected graph G which satisfies all of the following properties:

- G is connected and has at least 8 vertices
- G has *exactly* 4 distinct minimum spanning trees (MSTs)
- The edges of G have weights which are positive integers.

Explain why your graph G satisfies all of the aforementioned properties. **[6 marks]**

(b) Consider the following *divide-and-conquer* algorithm for finding a minimum spanning tree (MST) in an undirected, connected graph $G = (V, E)$:

> Algorithm: Consider any set $X \subseteq V$. Recursively, find minimum spanning trees T_1 and T_2 for the subgraphs induced by G on X and $V \setminus X$ respectively. The edges that we output in the MST of G are $T_1 \cup T_2 \cup \{e\}$ where e is the edge of minimum weight among all edges which have exactly one endpoint in X and one endpoint in $V \setminus X$.

Design a graph G which satisfies the following properties:

- The above algorithm does **not** find a MST for G
- G is connected and has at least 8 vertices
- The edge-weights in G are all distinct positive integers, i.e., no two edges of G have the same weight.

Explain why your graph G satisfies all of the aforementioned properties. **[10 marks]**
NB. Recall that if $G = (V, E)$ is an undirected graph and $Z \subseteq V$ then the graph induced by G on Z is the graph whose vertex set is Z and edge set is the set of all edges from E which have both endpoints in Z.

Question 3

(a) What is the definition of the class **NP**? **[4 marks]**

(b) Show that, if $L_0, L_1 \in \mathbf{NP}$ then also $L_0 \cup L_1 \in \mathbf{NP}$. **[6 marks]**

Question 4

Consider the Boolean function $f(a, b, c, d)$ returning 1 if and only if at least 2 of a, b, c, d are 1.

(a) Write a DNF computing $f(a, b, c, d)$ without using any negated literals. **[2 marks]**

(b) Consider the function $\bar{f}(a, b, c, d)$ returning 1 just if at most 1 of a, b, c, d are 1. Write out a CNF computing $\bar{f}(a, b, c, d)$, and justify your answer. **[4 marks]**

Hint: Notice that $\bar{f}(a, b, c, d)$ is the *negation* of $f(a, b, c, d)$.

End of Paper

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Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.