A36934

Calculators may be used in this examination provided they are not capable of being used to store alphabetical information other than hexadecimal numbers



UNIVERSITY OF BIRMINGHAM

School of Computer Science

Third Year Undergraduate

06-35308

LH Algorithms and Complexity

Main Summer Examinations 2025

Time allowed: 2 hours

[Answer all questions]

Note

Answer ALL questions. **There are four questions worth 20, 20, 30, 30 marks each for a total of 100 marks.** You can use (without proof) anything that was taught in the module (slides, lectures, weekly assignments).

Question 1

A preference list for a hospital (respectively student) is a ranking over the set of students (respectively hospitals). Throughout this question, the preference lists of hospitals must be *strict* and *complete*, i.e., each hospital ranks all the students, and there are no ties when a hospital has to consider ranking between two students. The same requirement of strict and complete preference lists also holds for those of the students.

Recall that a *matching* between n hospitals and n students is a pairing where each hospital is paired to exactly one student and vice versa. A matching M is *stable* if there is no hospital h and student s which form an unstable pair, i.e., satisfy all the following conditions

- h and s are not matched in M
- h prefers s to the student it is matched to in M
- s prefers h to the hospital it is matched to in M

In the module, we have discussed the Gale-Shapley algorithm (where hospitals make offers to students) which finds a stable matching. Recall that there is a choice in the Gale-Shapley algorithm in the *while* loop when deciding which free hospital to choose for making an offer to its highest ranked student that it has not made an offer to yet.

Please answer the following question: suppose you have 4 hospitals and 4 students. Design preference lists for which there is an execution of the Gale-Shapley algorithm (with hospitals making offers to students) where **exactly** 4 offers are made (in total) by the hospitals to the students. Describe an execution of the Gale-Shapley algorithm on your preference lists and state the 4 offers that the hospitals make to the students. Explain why these are the only 4 offers that are made in your execution of Gale-Shapley algorithm.

[20 marks]

Question 2

You want to spend n days in a new city. There are two possible choices of hotels: Premier Inn and Travelodge. For each $1 \leq i \leq n$, the cost of staying at the Premier Inn on day i is P_i and the cost of staying at the Travelodge on day i is T_i where P_i, T_i are given positive integers. The cost of moving from one hotel to the other hotel is 100. You can start your stay at any of the two hotels.

Solve the following three questions:

(a) Design an example with $n = 5$ where the optimum solution chooses to start at the hotel which has the minimum cost for staying the first day (and explain why this happens). **[5 marks]**

(b) Design an example with $n = 5$ where the optimum solution does not choose to start at the hotel which has the minimum cost for staying the first day (and explain why this happens). **[5 marks]**

(c) Design an $O(n)$ time algorithm, via dynamic programming or otherwise, which calculates the least possible expenditure of an n-day stay. Explain both the correctness and running time of your algorithm. **[10 marks]**

Question 3

The input of the SAT problem is a CNF formula with N variables and M clauses, and the question is whether there exists an assignment of variables which satisfies *all* the clauses. Recall that we have discussed in the module that the SAT problem is **NP**-hard.

(a) Show that there always exists an assignment of the variables in the SAT problem which satisfies **at least half** of the clauses. Design an $O(M \cdot N)$ time algorithm that finds such an assignment. **[14 marks]**

(b) Show that the following problem is **NP**-complete:
Half-SAT: Given a CNF formula with N variables and M clauses where M is even, is there an assignment of variables which satisfies **exactly half** of the M clauses, i.e, satisfies exactly $M/2$ clauses. **[16 marks]**

NB: For this question, you may assume that each clause is satisfiable by at least one assignment.

Question 4

In the 3-Hitting-Set problem, we are given a set $U = \{u_1, u_2, \ldots, u_n\}$ and a set $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ which contains subsets of U. Additionally we are given that for each $1 \leq i \leq m$, the set S_i contains exactly three distinct elements from U.

Design an algorithm which runs in $3^k \cdot n^{O(1)}$ time and finds if there exists a set $X \subseteq U$ such that $|X| \leq k$ and $|X \cap S_i| \neq \emptyset$ for each $1 \leq i \leq m$, i.e., X contains at least one element from each S_i. Explain the correctness of your algorithm and analyze its running time.

Hint: Recall that checking if there is a 2-Hitting-Set of size $\leq k$ is equivalent to checking if there is a vertex cover of size $\leq k$, and for the latter problem we have discussed (in the module) an $2^k \cdot n^{O(1)}$ time algorithm (based on branching) where n is the number of vertices in the graph. **[30 marks]**

This page intentionally left blank.

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.