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Calculators may be used in this examination provided they are not capable of being used to store alphabetical information other than hexadecimal numbers



UNIVERSITY OF BIRMINGHAM

School of Computer Science

Third Year Undergraduate

06-40088

LH Game Theory

Main Summer Examinations 2025

Time allowed: 2 hours

[Answer all questions]

Note

Answer ALL questions.

Question 1

(a) Identify all saddle points in the following zero-sum games

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

[10 marks]

(b) Calculate the maximin and minimax values (show all your working) for the zero-sum game

$$\begin{pmatrix} 2 & 3 & 1 & 0 & 1 \\ 4 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 & -7 \\ 2 & 3 & 1 & 0 & 1 \\ 2 & 3 & 1 & 4 & -3 \end{pmatrix}$$

Apply the minimax theorem. Does this game have an equilibrium? If yes, identify all equilibria. **[6 marks]**

(c) Identify all equilibria in the following (symmetric) games
Where applicable, distinguish cases for the real-valued parameter a.

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ a & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

[10 marks]

(d) Can you find a dominated strategy in the following bimatrix game? Is it a weakly or strictly dominated strategy?

$$\begin{pmatrix} (1,-1) & (3,-4) & (-2,4) & (4,-2) \\ (0,-1) & (0,-3) & (-3,0) & (2,-1) \\ (1,-1) & (1,-2) & (-1,-1) & (2,3) \end{pmatrix}$$

[6 marks]

(e) Construct a 3x3 zero-sum game with exactly 4 saddle points. **[4 marks]**

Question 2

Consider the (symmetric) game (a is a real-valued parameter)

$$\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

(a) Determine all pure strategy equilibria (depending on a). **[4 marks]**

(b) Can there be a mixed strategy equilibrium? Show all your working and reasoning. **[6 marks]**

(c) Derive the replicator equation for this game. Show all your working. **[6 marks]**

(d) Identify all fixed points of this equation. Discuss, which of these fixed points are relevant in this game. Which of the fixed points are stable? Where applicable, distinguish cases for a. **[4 marks]**

(e) Consider now the (symmetric) game (a is a real-valued parameter)

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & a \\ 0 & 1 & 0 \end{pmatrix}$$

Derive the replicator equations. Do NOT simplify terms at this stage. **[6 marks]**

(f) Assuming $a = 1$, derive for which value of b the replicator equations would have a fixed point at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. **[4 marks]**

(g) The Fermi update process is defined by the switching probability

$$p = \frac{1}{1 + \exp(-\beta(\pi_A - \pi_B))}.$$

Discuss the limits $\beta \to 0$ and $\beta \to \infty$. Sketch the switching probability (as a function of the payoff difference) and identify a relevant application area (including biology and socioeconomic sciences) for each case. **[4 marks]**

Question 3

Consider the zero-sum game described by the following payoff matrix with $a, b > 0$:

$$A = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}.$$

(a) For what values of the payoffs a, b, is the Nash equilibrium (NE) an Evolutionarily Stable Strategy (ESS)? Explain your answer. **[10 marks]**

(b) Derive the ODE corresponding to the replicator dynamics for this game. **[5 marks]**

Hint: recall that the ODE of the replicator dynamics for strategy i is: $\dot{x}_i = x_i((Ax)_i - x^\top Ax)$.

(c) Are there values of the payoffs a, b for which the Brown-von Neumann-Nash (BNN) dynamics exhibit the same behaviour as the replicator dynamics calculated in part (b)? Calculate the equilibria for these payoffs and motivate your answer using a stability argument. **[15 marks]**

Hint: recall that the ODE of the BNN dynamics for strategy i is: $\dot{x}_i = \hat{a}_i(x) - x_i \sum_{j=1}^{2} \hat{a}_j(x)$, where $\hat{a}_i(x) = [(Ax)_i - x^\top Ax]_+$.

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Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.